Quantum diffusion dynamics in nonlinear systems: A modified kicked-rotor model

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Using a simple method analogous to a quantum rephasing technique, a simple modification to a paradigm of classical and quantum chaos is proposed. The interesting quantum maps thus obtained display remarkably rich quantum dynamics. Emphasis is placed on the destruction of dynamical localization without breaking periodicity, unbounded quantum anomalous diffusion in integrable systems, and transient dynamical localization. Experimental realizations of this work are also discussed.

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Studies of dynamical models with rich dynamical features have exposed the many faces of quantum dynamics in nonlinear systems [1–7]. The so-called quantum kicked rotor (QKR) [1] is the most studied and has motivated the rare exercise of synthesizing a dynamical model in many cold-atom laboratories worldwide. More insights into quantum coherence and quantum-classical correspondence in nonlinear systems are expected in the near future. Emerging new applications of QKR studies include nondestructive stopping of quantum-information transfer [8], efficient simulation of a many-body bath [9], and the observation of metal-insulator transitions [10].

When a noteworthy quantum phenomenon is observed from a nonlinear model, its universality vs peculiarity should be evaluated with caution by considering, for example, variants of that model. This is often an involved task. Take the celebrated "dynamical localization" (DL) [11] in the QKR [1] and many QKR variants [12] as a dramatic example of quantum suppression of classical diffusion. Fishman *et al.* [11] showed that the spectrum of QKR can be mapped to that of a one-dimensional Anderson model with diagonal pseudorandomness, thereby rationalizing the observation of saturated quantum diffusion dynamics, albeit unbounded classical chaotic diffusion. Casati *et al.* showed that the DL can be well described by a theory of banded random matrices [13]. However, on a deep level, the nature of DL in a QKR and its variants is still unclear. For example, for generic system parameters, is the pseudorandomness responsible for DL intrinsically connected with the underlying classical chaos? Further, can the DL in a QKR be coherently removed while maintaining the periodicity and smoothness of the kicking field?

In this paper we stimulate fundamental studies of quantum coherence in unbounded nonlinear systems by proposing a physically achievable variant of the QKR with a well-defined classical limit. Our work offers additional opportunities for identifying peculiarities in the QKR and exposes more interesting aspects of quantum nonlinear dynamics that may occur in a kicked-rotor system. In particular, we show (a) that DL can be absent in a periodically kicked chaotic system modified from the QKR, even with those generic system parameters that otherwise lead to strong DL in the QKR, (b) that unbounded quantum anomalous diffusion may also generically occur in a classically integrable variant of the QKR, and (c) that interesting crossover behavior can occur in the quantum diffusion dynamics at very large time scales, including a phenomenon called transient DL. Because none of these results applies to the QKR and its variants known to date, the hope is that our model can motivate more fundamental studies of quantum diffusion dynamics and place known theories (e.g., semiclassical theory and spectrum statistics theory) under new tests. As further discussed below, our model also displays some features that are more or less analogous to some properties of the kicked-Harper model [3–5], thus encouraging the search for a potential link between modified kicked-rotor models and the kicked-Harper model. In addition, the possibility of a new generation of experimental studies of quantum diffusion dynamics is also discussed.

The QKR Hamiltonian in scaled dimensionless variables can be written as $H_{QKR}=\hat{L}^2/2+k\cos(\theta)\Sigma_N\delta(t-N)$, where $\hat{L}$ and θ ($\theta\in[0,2\pi]$), the angular momentum operator and the angle variable, form a conjugate pair with $[\theta,\hat{L}]=i\tau$. Hence τ is the effective Planck constant here. N is the number of kicks. The basis states of the QKR Hilbert space, denoted by $|m\rangle$, satisfy $\hat{L}|m\rangle=m\tau|m\rangle$ and the periodic boundary condition. In the same Hilbert space, we now adapt H_{QKR} to construct the model below:

$$H=\begin{cases}-\dfrac{\hat{L}^2}{2}+k\cos(\theta)\displaystyle\sum_N\delta(t-N), & 2j\le N<2j+1, \quad (1)\\[2ex] \dfrac{\hat{L}^2}{2}+k\cos(\theta)\displaystyle\sum_N\delta(t-N), & 2j+1\le N<2j+2, \quad (2)\end{cases}$$

where j is an arbitrary integer. Clearly, H differs from H_{QKR} only in the term $-\hat{L}^2/2$ for $2j\le N<2j+1$. Because this difference allows for a negative contribution of the kinetic energy term, at first sight H seems to be an unphysical model. This is not the case, if we note the following three facts. First, H generates a well-behaved quantum map $\hat{U}$ (the propagator associated with every two time steps). That is,

$$\hat{U}=\hat{W}\exp[-ik\cos(\theta)], \quad (3)$$

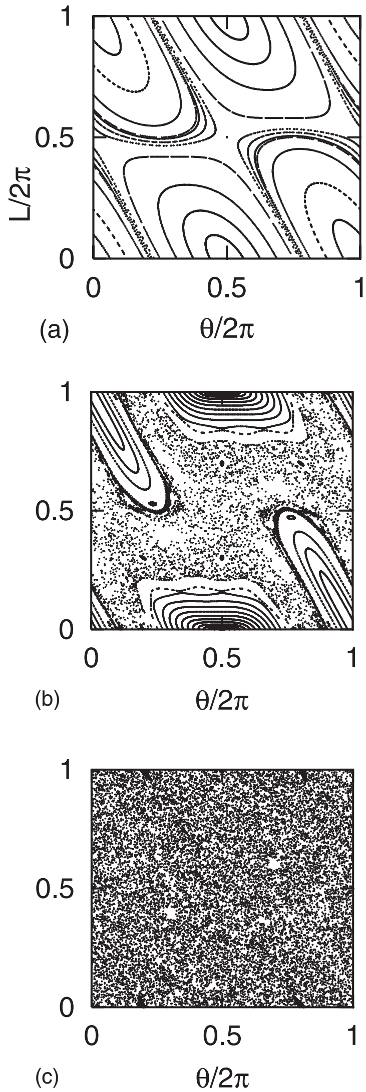

FIG. 1. Classical phase space structure of H [see Eqs. (1) and (2)]. κ=(a) 0.5, (b) 2.0, and (c) 4.0. H with $\kappa \leq 0.5$ is predominantly integrable and H with $\kappa \geq 4.0$ is predominantly chaotic.

$$\hat{W} = \exp(i\hat{L}^2/2\tau)\exp[-ik\cos(\theta)]\exp(-i\hat{L}^2/2\tau). \quad (4)$$

Second, as discussed below, the quantum map $\hat{U}$ can be physically realized in, for example, a Heisenberg spin chain in external fields. Third, the classical limit of $\hat{U}$, obtained by letting $\tau \to 0$ and keeping $\tau k = \kappa$ as a constant, is also well defined and can be written as follows:

$$L_{2j+1} = L_{2j} + \kappa \sin(\theta_{2j}), \quad \theta_{2j+1} = \theta_{2j} - L_{2j+1}, \quad (5)$$

$$L_{2j+2} = L_{2j+1} + \kappa \sin(\theta_{2j+1}), \quad \theta_{2j+2} = \theta_{2j+1} + L_{2j+2}, \quad (6)$$

where (L_N, θ_N) denotes a classical phase space point right before the Nth kick. The associated classical phase space structure for different values of κ is shown in Fig. 1, displaying predominantly integrable, mixed, and predominantly chaotic dynamics as κ increases, a trend analogous to that in the QKR.

One noteworthy property of H as a variant of the QKR is as follows. Equation (4) indicates that $\hat{W}$ is connected with $\exp[-ik\cos(\theta)]$ by a unitary transformation. Hence the spectrum of $\hat{W}$, identical with that of $\exp[-ik\cos(\theta)]$, is continuous. As such, the quantum map $\hat{U} = \hat{W}\exp[-ik\cos(\theta)]$ is identified as a product of two unitary operators with identical and continuous spectra (but with different eigenfunctions). It is this property, with a potential mathematical interest as well, that makes a clear difference from the original QKR and its variants studied previously [12]. Indeed, without our modification to the QKR, $\hat{W}$ would have taken the form $\exp[i(\hat{L}^2/2\tau)]\exp[-ik\cos(\theta)]\exp[i(\hat{L}^2/2\tau)]$, which in general has a discrete spectrum.

To further analyze the quantum map operator $\hat{U}$, consider first

$$W_{n,m} \equiv \langle n|\hat{W}|m\rangle = (-i)^{m-n} J_{m-n}(k)\exp[i\tau(n^2 - m^2)/2], \quad (7)$$

where J_{m-n} is the Bessel function of order $(m-n)$. The diagonal terms $W_{m,m}$ are seen to be the constant $J_0(k)$. The off-diagonal elements contain pseudorandom factors $\exp(-i\tau m^2/2)$ [11] and $\exp(i\tau n^2/2)$. However, these two factors tend to cancel each other so long as $|m-n|$ is small. This is analogous to a quantum rephasing or refocusing technique, i.e., the pseudorandom dynamical phase $\exp(i\tau n^2/2)$ associated with $\hat{L}^2/2$ is compensated for by the dynamical phase $\exp(-i\tau m^2/2)$ associated with the term $-\hat{L}^2/2$. For large $|n-m|$, $W_{n,m}$ becomes negligible because $J_{m-n}(k)$ exponentially decays with increasing $|m-n|$. Indeed, in the deep quantum regime limit (small-k limit), only the "nearest-neighbor" coupling contributes, giving rise to the quasiperiodic amplitude $W_{m,m+1} = -iJ_1(k)\exp[-i\tau(2m+1)/2]$.

We are now ready to analyze the matrix elements of $\hat{U}$, given by

$$U_{n,m} \equiv \langle n|\hat{U}|m\rangle = \sum_l (-i)^{m-l} J_{m-l}(k) W_{n,l}. \quad (8)$$

Using the deep quantum regime approximation, one has

$$U_{m,m} \approx J_0^2(k) - 2J_1^2(k)\cos(\tau m)\exp(-i\tau/2), \quad (9)$$

$$U_{m,m+1} \approx -iJ_0(k)J_1(k)\{1 + \exp[-i\tau(2m+1)/2]\}, \quad (10)$$

$$U_{m,m+2} \approx -J_1^2(k)\exp[-i\tau(2m+1)/2], \quad (11)$$

with all other matrix elements containing higher-order Bessel functions neglected. Evidently then, the diagonal term $U_{m,m}$ is given by a constant plus quasiperiodic oscillations; and the transition amplitudes $U_{m,m+1}$ and $U_{m,m+2}$ have quasiperiodic structures as well. In particular, the nearest-neighbor transition strength $|U_{m,m+1}|$ may reach almost zero quasiperiodically, implying interesting dynamical features observable only at long time scales (see below). Because neither the diagonal terms nor the main off-diagonal terms are pseudorandom for any value of τ, it is evident that a banded random matrix theory, very successful in describing the standard QKR, can no longer model the quantum map $\hat{U}$ as a simple

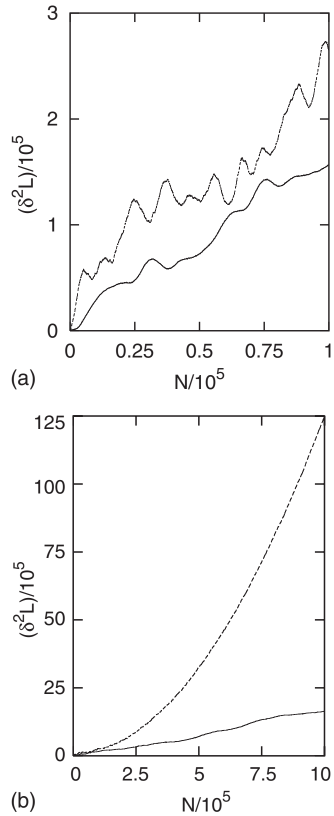

FIG. 2. Unbounded quantum diffusion dynamics associated with H for 10^5 kicks in (a) and 10^6 kicks in (b). The initial state is given by $|0\rangle$. Solid and dashed lines are for two different sets of k and τ. The classical dynamics is predominantly chaotic [see Fig. 1(c)] and displays linear diffusion.

variant of the QKR. Likewise, the Anderson localization approach will not apply, either. Nonetheless, with the pseudorandomness in U_{nm} thus removed, $\hat{U}$ can still have a chaotic classical limit [see Fig. 1(c)]. This makes it clear that the pseudorandomness previously emphasized in the matrix elements of the QKR quantum map does not have an inherent connection with classical chaos.

To shed light on and motivate formal studies of the spectrum of $\hat{U}$ as a product of two unitary operators of identical and continuous spectrum, we briefly describe below representative and informative numerical results from extensive numerical experiments, leaving detailed discussions to future publications. Consider first two examples of quantum diffusion dynamics shown in Fig. 2(a), both with a chaotic classical limit. The dynamics is displayed in terms of the time dependence of the variance in $\hat{L}$ (denoted $\delta^2 L$) for as long as $N=10^5$ kicks. We consider only cases with irrational τ/π so that the dynamics will be unrelated to the well-known nongeneric quantum resonance diffusion in the QKR. In particular, in the first example (solid line), $\kappa=4.0$, $\tau=\pi/(2+\sigma_g)$, where $\sigma_g=(\sqrt{5}-1)/2$ is chosen as a "golden mean" number such that τ/π is as irrational as possible. In the second example (dashed line), $\kappa=5.0$, $\tau=2.0$ (a convenient choice of τ, but still irrational with π). For these parameters, the standard QKR will clearly display DL within about 10^2 kicks with $\delta^2 L$ saturated around 10^2. In clear contrast to the QKR results, here [Fig. 2(a)] no sign of saturation in the diffusion dynamics is observed. Instead, the variance in $\hat{L}$ (note the 10^5 scale) displays an unbounded and somewhat linear time dependence, with an average diffusion rate a few times smaller than the classical linear diffusion rate.

The results in Fig. 2(a) have been checked by running independent codes on different machines. To further confirm the observed unbounded diffusion, the quantum dynamics is examined over a much longer time scale, i.e., over 10^6 kicks. The associated results are shown in Fig. 2(b), clearly indicating that, in a simple variant of the QKR, DL can be absent in the fully coherent quantum mapping dynamics for irrational values of τ/π. Significantly, the first example (solid line) maintains its somewhat linear diffusion throughout, whereas the second example (dashed line) is seen to undergo a crossover toward a simple and evident polynomial time dependence of $\delta^2 L$. An excellent log-log linear fit to the diffusion dynamics (dashed line) for $2.5 \leq N/10^5 \leq 10$ gives $\delta^2 L \sim N^{1.92}$ (essentially a quadratic time dependence) for that regime. This unexpected crossover is truly surprising because it occurs at very long times ($>10^5$ kicks). It implies the complex nature of the spectrum of the quantum map $\hat{U}$. That is, as we increase the time scale new spectral properties at a finer scale start to manifest themselves. In addition, the coexistence of the almost quadratic diffusion and the somewhat linear diffusion for different k and τ suggests a strong sensitivity of the spectrum to variations in the system parameters. By tuning k and τ it should be possible to observe intermediate cases as examples of unbounded quantum anomalous diffusion in a modified kicked-rotor system.

Can unbounded quantum diffusion also occur if κ is sufficiently small such that the underlying classical dynamics is predominantly integrable and the classical diffusion rate is essentially zero due to the regular phase space structure? To that end we consider two computational examples for $\kappa=0.5$, with the same initial state as in Fig. 2. The classical regular phase space structure is shown in Fig. 1(a). Results for two values of the effective Planck constant, i.e., $\tau=\pi/(2+\sigma_g)$ (solid line) and $\tau=1.0$ (dashed line) are shown in Fig. 3. Unexpectedly, the quantum diffusion dynamics is quite analogous to those in Fig. 2, despite the fact that the classical $\delta^2 L$ quickly saturates at a value of 1.87×10^0. Even more significantly, a crossover to superdiffusion is also observed in the dashed-line case. A log-log linear fit to the dashed line in Fig. 3(b) for $2.5 \leq N/10^5 \leq 10$ yields $\delta^2 L \sim N^{1.72}$, a clear sign of quantum anomalous diffusion.

The results in Fig. 3 should not be regarded as a violation of the quantum-classical correspondence principle, because the effective Planck constant τ used is not small and the time scale under investigation is very large. To gain more insights we also examined cases with much smaller τ, finding again unbounded quantum diffusion with regular classical dynamics. However, in these cases the diffusion is much slower and excellent quantum-classical agreement becomes clearly observable for a short period. Interestingly, as the quantum result gradually deviates from the classical, a high concentration of quantum amplitude is built up along a web of separatrices associated with the unstable fixed points in each phase space cell [$\theta=\pi$, $L=(2j+1)\pi$] [14]. This is analogous

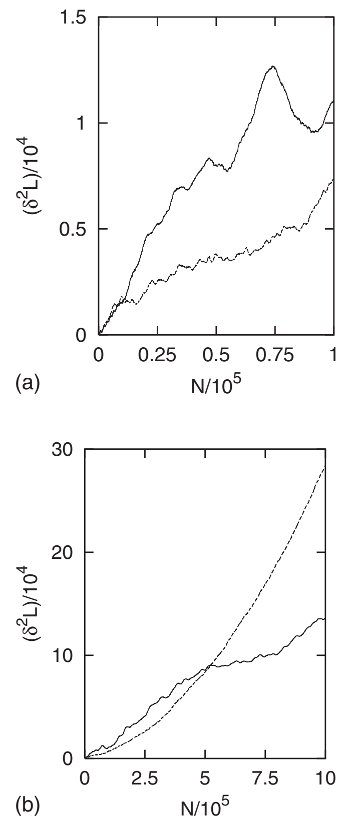

FIG. 3. Unbounded quantum diffusion dynamics in the case of predominantly integrable classical dynamics [see Fig. 1(a)] that yields a zero classical diffusion rate. Solid and dashed lines are for two different values of the effective Planck constant, for 10^5 kicks in (a) and for 10^6 kicks in (b).

to the so-called quantum scars in chaotic systems. Detailed results will be presented elsewhere [14].

Along with the late crossover observed in Figs. 2 and 3, another interesting finding can be called transient DL in a modified kicked-rotor model. Specifically, consider two computational examples in Fig. 4 that show clear saturation behavior of $\delta^2 L$ for as long as $\sim 10^6$ kicks. The saturated variance averaged over 10^6 kicks is much smaller than the typical values of $\delta^2 L$ shown in Figs. 2(b) and 3(b). In addition to this saturation behavior, which is characteristic of DL, the excitation line shape (not shown) is also studied, often displaying exponential forms (as in DL) plus quasiperiodic structures that match well the quasiperiodic appearance of almost zero matrix elements $U_{m,m+1}$ [see Eq. (10)]. At this point it becomes very tempting to conclude that true DL is then observed for these system parameters. Surprisingly, this is not the case, as already indicated by Fig. 4. The DL signatures observed for the first 10^6 kicks do not persist for even larger time scales. Instead, the saturation behavior later breaks down and the quantum diffusion resumes. Such transient DL behavior observed in a modified kicked-rotor model implies once again that DL is a truly subtle quantum phenomenon.

The numerical results reported here open up many theoretical questions. In particular, does the transient DL imply the existence of a mobility edge for $\hat{U}$? How can the spectrum of $\hat{U}$ be characterized in connection with the quantum diffusion dynamics? In attacking these questions, previous studies of the kicked-Harper model [3–5,7] (which also displays unbounded quantum diffusion with a zero classical diffusion rate as well as regular classical phase space structure) should be helpful. Indeed, by directly truncating the quantum map $\hat{U}$ to a finite-dimensional Hilbert space, the spectrum of $\hat{U}$ is seen to display a Hofstadter butterfly structure that appears analogous to the kicked-Harper model [14]. The hope is then that our model here can provide a bridge between the QKR and the kicked-Harper model.

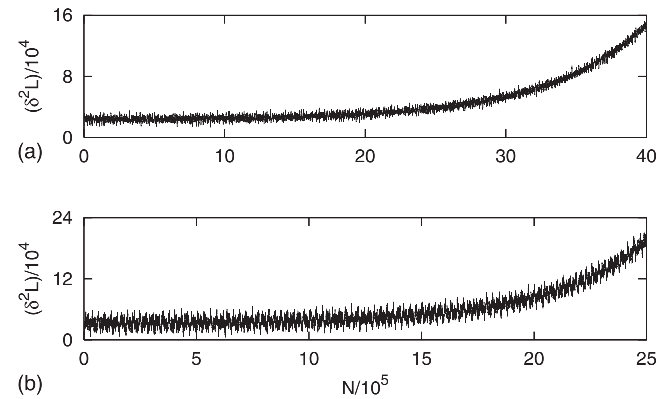

FIG. 4. Two examples of quantum transient DL observed in the quantum diffusion dynamics associated with the quantum map $\hat{U}$. In (a) $\kappa=2.0$, $\tau=2.0$; and in (b) $\kappa=10.0$, $\tau=10.0$. The saturation of $\delta^2 L$ during the first 10^6 kicks is evident, but quantum diffusion then resumes for larger time scales.

We now briefly discuss how to physically realize the quantum map $\hat{U}$. Consider first a spin-chain realization of the standard QKR [8,15], obtained by subjecting a Heisenberg spin chain to a parabolic kicking magnetic field. The Hamiltonian is given by

$$H_s = -\frac{J}{2}\sum_n \sigma_n \cdot \sigma_{n+1} - B\sum_n \sigma_n^z + \sum_n \sigma_n^z \frac{Cn^2}{2}\sum_N \delta(t-N), \tag{12}$$

where $\sigma \equiv (\sigma^x, \sigma^y, \sigma^z)$ are the Pauli matrices, J is the spin-spin interaction constant, n denotes the index of spins, B is a constant field lifting the degeneracy of the system, and $(Cn^2/2)\Sigma_N \delta(t-N)$ describes a parabolic kicking field. Upon linking the index of the spins with the quantum number m, and linking the spin-wave quasimomentum with the angular variable θ, the quantum propagator of the spin chain system here with one spin flipped is found to be equivalent to that for the standard QKR. As such, the quantum map $\hat{U}$ can be easily realized if we reverse the parabolic kicking field after every kick. That is, the "negative kinetic energy" term in H can be realized by a reversed parabolic kicking potential. Extending this approach, one may consider a simpler tight-binding system subject to a parabolic kicking field, with the Hamiltonian

$$H_t = \frac{J}{2}\sum_n (|n\rangle\langle n+1| + \text{H.c.}) + \sum_n \frac{Cn^2}{2}|n\rangle\langle n|\sum_N \delta(t-N). \tag{13}$$

It can be proved that this tight-binding system yields the quantum map $\hat{U}$ if the sign of C is reversed after every kick. Experimental studies can hence be carried out in such systems as well. The results should be also of general interest for "coherent control," especially in the context of quantum phase control in classically chaotic systems [16].

To conclude, by a simple modification to the well-known kicked rotor model, we have proposed an interesting class of quantum maps with their rich properties remarkably different from those of previous modified kicked-rotor models. Three important lessons are that (a) even for an integrable variant of QKR, classical phase space structure is not reliable for qualitative understanding of the quantum diffusion dynamics, (b) DL is not directly related to classical chaos, and can be absent in a class of periodically kicked rotor models, regardless of the system's effective Planck constant, and (c) in the absence of DL, the quantum diffusion dynamics in a modified kicked-rotor model can still yield transient DL for a very large time scale. The numerical results briefly reported here, together with the details to be presented elsewhere [14], will attract more interest in the further understanding of quantum dynamics in unbounded nonlinear systems.

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